Local analysis of near fields in acoustic scattering

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Abstract

We compute near fields using boundary integral equation methods for 2D acoustic scattering by an obstacle with an analytic boundary. Accurate computation of near fields is needed for optical scattering by nanostructures and for other related problems. A classical method to approximate the solution everywhere consists of using the same quadrature rule (Nyström method) used to solve the underlying boundary integral equation. It is established that this method incurs an $O(1)$ error for a fixed number of quadrature points. Our goal is, for a fixed number of quadrature points and without using high-order Nyström methods, to develop a method to address this $O(1)$ error. Similar to numerical method for approximating singular integrals, we subtract from the associated kernel the asymptotic expansion that captures the nearly singular behavior.

Keywords: boundary integral equation, close evaluation, local analysis

1 Problem setting

We consider the following scattering problem by a sound-soft obstacle $D \subset \mathbb{R}^2$ with ∂D an analytic, closed curve:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \tag{1a}$$

$$u = f \quad \text{on } \partial D, \tag{1b}$$

$$\partial_r u - \mathrm{i}ku = o(r^{-1/2}), \quad r \to \infty, \tag{1c}$$

where k denotes the wavenumber and f is an analytic function that gives the field incident on the obstacle. The solution of (1) may be represented as a single- and double-layer potential (see [5]): for all $x \in \mathbb{R}^2 \setminus \bar{D}$,

$$u(x) = \int_{\partial D} \left[\partial_{n_y} G(x,y) - ikG(x,y)\right] \mu(y) \mathrm{d}\sigma_y. \tag{2}$$

The fundamental solution of (1a) is

$$G(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|), \tag{3}$$

where $H_0^{(1)}$ is the Hankel function of first kind, and the density μ satisfies the boundary integral equation for all $y' \in \partial D$,

$$\frac{1}{2}\mu(y') + \int_{\partial D} \partial_{n_y} G(y',y)\mu(y)\mathrm{d}\sigma_y - ik\int_{\partial D} G(y',y)\mu(y)\mathrm{d}\sigma_y = f(y'). \tag{4}$$

Since ∂D is a closed, analytic curve, and G exhibits a log-singular behavior, (4) can be solved numerically with spectral accuracy using Kress Nyström method [5, Chapter 12] (for Laplace we consider the periodic trapezoid rule [1]). Using the same Nyström method to evaluate (2) incurs an $O(1)$ error for points in $\mathbb{R}^2 \setminus \overline{D}$ that are close to ∂D. This is due to the fact that the kernel $K := \partial_{n_y} G - ikG$ is nearly singular, in the sense that K is sharply peaked when $|x-y| \to 0^+$, and will not be well resolved for fixed quadrature points. In fact, the error made in evaluating (2) exhibits a boundary layer with thickness $O(1/N)$ where N is the number of quadrature points, leading to a $O(1)$ error as x approaches ∂D [1]. It is necessary to accurately predict these near fields for optical scattering by nanostructures, for instance in plasmonics.

2 Local analysis and numerical results

To address the $O(1)$ error associated with the near-field evaluation problem, we treat nearly singular integrals in a similar fashion to methods developed for singular integrals [3]. We subtract K^{ns} the nearly singular behavior of the kernel K appearing in (2) and write the solution, for all $x \in \mathbb{R}^2 \setminus \overline{D}$, as

$$u(x) = \int_{\partial D} (K(x,y) - K^{ns}(x,y))\mu(y)\mathrm{d}\sigma_y + \int_{\partial D} K^{ns}(x,y)\mu(y)\mathrm{d}\sigma_y. \tag{5}$$

In (5) the first integral is smooth, and therefore easier to approximate, whereas the second one is evaluated analytically. K^{ns} is found as a linear combination of the asymptotic expansions of the single- and double-layer potentials. Let

δ denote the separation distance between the evaluation point x in $\mathbb{R}^2 \setminus \overline{D}$ and the boundary ∂D: then $x = y^* + \delta n_{y^*}$ with $y^* \in \partial D$. Defining $\mathtt{Y} := \frac{y^*-y}{\delta}$, one can rewrite K as a non-uniform expansion $K(x,y) = K(\delta\mathtt{Y}, \delta)$, then K^{ns} is found as the inner expansion of $K(\delta\mathtt{Y},\delta)$ when $\mathtt{Y} \to 0^+$. It is well-known that G has the same singular behavior as G^L, the fundamental solution of Laplace's equation [5]:

$$G = G^L + \text{cst} + O(\delta^2 \log \delta), \qquad (6)$$

with $\text{cst} := \dfrac{i}{4} - \dfrac{1}{2\pi}\left(\log \dfrac{k}{2} + C\right)$, and C denoting Euler's constant. Therefore, the leading order of K^{ns} can be found as the inner expansion of $K^L := \partial_{n_y} G^L - ik(G^L + \text{cst})$, with

$$\begin{aligned} G^L(\delta\mathtt{Y}, \delta) = &-\frac{1}{2\pi}\log\delta \\ &-\frac{1}{4\pi}\log(1 + |\mathtt{Y}|^2 - 2\delta n_{\mathtt{Y}^*}\cdot\mathtt{Y}), \end{aligned} \qquad (7)$$

$$\partial_{n_y} G^L(\delta\mathtt{Y},\delta) = -\frac{1}{2\pi\delta}\frac{n_{\mathtt{Y}}\cdot\mathtt{Y} + n_{\mathtt{Y}}\cdot n_{\mathtt{Y}^*}}{1 + |\mathtt{Y}|^2 - 2n_{\mathtt{Y}^*}\cdot\mathtt{Y}}. \qquad (8)$$

Using the parametrization $\mathtt{Y}(t) = \dfrac{y(t^*) - y(t)}{\delta}$, $t, t^* \in [0, 2\pi]$, one can express K^{ns} as a rational trigonometric function of the form

$$\begin{aligned} K^{ns}(t,t^*;\delta) = &\frac{A_0 + A_1\cos(t-t^*)}{1 + B_1\cos(t-t^*)} \\ &- ik\left(\log(C_0 + C_1\cos(t-t^*)) + D\right), \end{aligned} \qquad (9)$$

where A_0, A_1, B_1, C_0, C_1, D are constants, in particular depending on δ and the curvature of the boundary at y^*. The integral operator with K^{ns} can be computed spectrally using its Fourier series representation [4]. We can then compute (5) efficiently and accurately.

Results in Fig. 1 show a gain of at least 3 orders of precision close to boundary. Electrostatic cases ($k = 0$) have shown a gain of at least 6 orders. Since $K \sim K^L$ is valid for $k|x-y|$ sufficiently small, we perform a subwavelength correction (i.e. $k\delta \ll 1$). A new scaling taking into account k will be required to tackle high frequency scattering problems [1].

3 Future works

These results show the advantage of incorporating asymptotic analysis into the numerical evaluation of near-fields. The asymptotic Nyström method with sub-wavelength correction can be improved further using the outer expansion of K (i.e. $\mathtt{Y} \to \infty$) and using a subtraction method applied to the density μ [3]. Further details will be given in [2]. Extensions to 3D configurations will be considered, and we will apply these techniques for Stokes problems and scattering problems in plasmonic structures.

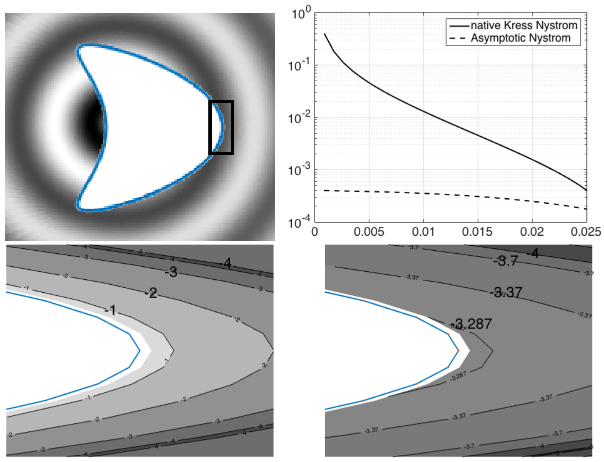

Figure 1: Top left: real part of the solution of (2) given by $u(x) = \frac{i}{4}H_0^{(1)}(k|x-x_0|)$, with $k = 5$, $x_0 = (-0.8, 0.2) \in D$, with $N = 300$. Top right: error (log-scale) with respect to δ for $t^* = \frac{29}{30}2\pi$. Bottom: Contour errors (log-scale) for (2) at the rectangle indicated in the top left figure, using native Nyström method (left), and the asymptotic Nyström method (right).

References

[1] A. H. Barnett, Evaluation of layer potentials close to the boundary for Laplace and Helmholtz problems on analytic planar domains, *SIAM J. Sci. Comput.* **36** (2014), pp. A427–A451.

[2] C. Carvalho, S. Khatri, A. D. Kim, Local analysis for close evaluation of layer potentials, *in preparation.*

[3] L. M. Delves, J. L. Mohamed, *Computational Methods for Integral Equations*, Cambridge University Press, 1985.

[4] J. F. Geer, Rational trigonometric approximations using Fourier series partial sums, *Journal of Scientific Computing*, **10** (1995), pp. 325–356.

[5] R. Kress, Boundary integral equations in time-harmonic acoustic scattering, *Math. Comput. Model.* **15** (1991), pp. 229–243.