**Plasmonic waveguides: T-coercivity approach for Maxwell's equations**

**A.-S. Bonnet-Ben Dhia**[1], **C. Carvalho**[2,*], **P. Ciarlet Jr.**[3]
[1]POEMS, UMR 7231 CNRS-INRIA-ENSTA, Palaiseau, France
[2]POEMS, UMR 7231 CNRS-INRIA-ENSTA, Palaiseau, France
[3]POEMS, UMR 7231 CNRS-INRIA-ENSTA, Palaiseau, France
*Email: camille.carvalho@ensta-paristech.fr

**Suggested Scientific Committee Members:**
Rabia Djellouli, Nabil Gmati, Martin Costabel

**Abstract**

We look for the electromagnetic guided modes in a closed waveguide made of layers of materials characterized by real permittivities of opposite signs: we will consider a dielectric and a metal at optical frequencies. Due to this sign-changing permittivity, self-adjointness can be compromised. However, under some conditions, it can be recovered thanks to the T-coercivity approach. The T-coercivity theory has been extensively developed for scalar problems with sign-changing coefficients, then extended to Maxwell 2D (no dependence in one direction) and Maxwell 3D. We extend these results to our case, referred to as the 2.5D case. When self-adjointness is ensured, with the adapted functional framework, and for a chosen wavenumber, we can prove resolvent compactness. Then we can derive error estimates for the approximation of eigenvalues and the guided modes using edge elements.

**Keywords:** Maxwell's equations, sign-changing permittivity, waveguide, T-coercivity, eigenvalue approximation

## 1 Problem setting

Let a domain $D := \{(x, y, z) := (\mathbf{x}, z) \in \Omega \times \mathbb{R}\}$ of section $\Omega \subset \mathbb{R}^2$, such that $\Omega := \Omega_\mathrm{d} \cup \Omega_\mathrm{m}$: $\Omega_\mathrm{d} \times \mathbb{R}$ is a homogeneous domain of permittivity $\epsilon_\mathrm{d} > 0$, permeability $\mu_\mathrm{d} > 0$, and $\Omega_\mathrm{m} \times \mathbb{R}$ a metal inclusion characterized by $\epsilon_\mathrm{m} < 0$, and $\mu_\mathrm{m} > 0$. Assume that $\Omega$ is simply connected with Lipschitz connected boundary, and define the interface $\Sigma := \overline{\Omega_\mathrm{d}} \cap \overline{\Omega_\mathrm{m}}$.

We look for the guided modes for the electromagnetic field $(\underline{\mathbf{E}}, \underline{\mathbf{H}})$, that is solutions of Maxwell's equations of the form:

$$(\underline{\mathbf{E}}, \underline{\mathbf{H}})(\mathbf{x}, z, t) = (\mathbf{E}, \mathbf{H})(\mathbf{x}) e^{i(\beta z - \omega t)},\ \omega, \beta \in \mathbb{R}, \tag{1}$$

where $\omega \neq 0$ is the frequency, and $\beta$ the axial wavenumber. It is well-known that, in particular for the unknown $\underline{\mathbf{H}}$, using (1), we can reduce the system into a 2D problem parametrized by $\beta$ that involves the three components of $\mathbf{H}$. We define new operators indexed by $\beta$ ($\mathbf{rot}_\beta$ and $\mathrm{div}_\beta$) which are simply a rewriting of the classical operators taking into account (1). Then we get:

$$\left| \begin{array}{l} \text{Find } \mathbf{H} \in \mathbf{W}(\Omega) \text{ such that:} \\ \mathbf{rot}_\beta \left( \dfrac{1}{\epsilon} \mathbf{rot}_\beta \mathbf{H} \right) - \omega^2 \mu \mathbf{H} = 0 \quad \text{in } \Omega \\ \dfrac{1}{\epsilon} \mathbf{rot}_\beta \mathbf{H} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \end{array} \right. \tag{2}$$

with $\mathbf{n}$ the unit outward normal of $\Omega$, $\epsilon$ and $\mu$ two piece-wise constant functions gathering the permittivity and the permeability of the two materials. Finally for all $\mathbf{F} := (\overrightarrow{F_\perp}, F_z)^t$ define

$$\mathbf{W}(\Omega) = \{\mathbf{F} / \overrightarrow{F_\perp} \in \overrightarrow{H}(\mathrm{rot}; \Omega),\ F_z \in H^1(\Omega)\}$$

where $\overrightarrow{H}(\mathrm{rot}; \Omega) := \{\overrightarrow{F_\perp} := (F_x, F_y)^t \in L^2(\Omega)^2 / \mathrm{rot}\overrightarrow{F_\perp} := \partial_x F_y - \partial_y F_x \in L^2(\Omega)\}$. One can show that a solution $\mathbf{H}$ of (2) also satisfies the conditions

$$\mu \mathbf{H} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega,\ \mathrm{div}_\beta(\mu \mathbf{H}) = 0 \text{ in } \Omega. \tag{3}$$

To look for the guided modes, we interpret Problem (2) as an eigenvalue problem: for a chosen $\beta \in \mathbb{R}$, find $(\mathbf{H}, \omega^2) \in \mathbf{W}(\Omega) \setminus \{0\} \times \mathbb{C}$ satisfying (2). Since $\epsilon$ is sign-changing, well-posedness for the forward problem (that is (2) with some data at the right-hand side) is not automatically guaranteed. First we need to ensure existence and uniqueness of this forward problem then, when it is satisfied, we can study the eigenproblem, and tackle the approximation of the eigenvalues.

## 2 Self-adjointness

As $\mathbf{W}(\Omega)$ is not compactly embedded in $\mathbf{L}^2(\Omega)$, we will work in a subspace that takes into account the divergence free condition (3):

$$\mathbf{V}_T(\beta;\mu;\Omega) = \{\mathbf{F} \in \mathbf{W}(\Omega)/ \\ \mu\mathbf{F}\cdot\mathbf{n} = 0 \ \partial\Omega,\ \mathrm{div}_\beta(\mu\mathbf{F}) = 0 \ \Omega\},$$

and which is compactly embedded in $\mathbf{L}^2(\Omega)$. We can prove that solving (2) is equivalent to:

$$\left| \begin{array}{l} \text{Find } \mathbf{H} = (\overrightarrow{H_\perp}, H_z)^t \in \mathbf{V}_T(\beta;\mu;\Omega) \text{ such that:} \\ a(\mathbf{H},\mathbf{H}') + c(\beta;\mathbf{H},\mathbf{H}') = \omega^2(\mu\mathbf{H},\mathbf{H}') \\ \forall \mathbf{H}' \in \mathbf{V}_T(\beta;\mu;\Omega), \end{array} \right. \tag{4}$$

with

$$\begin{aligned} a(\mathbf{H},\mathbf{H}') &:= a_\perp(\overrightarrow{H_\perp}, \overrightarrow{H_\perp}') + a_z(H_z, H_z') \\ &= \int_\Omega \frac{1}{\epsilon}\mathrm{rot}\overrightarrow{H_\perp}\overline{\mathrm{rot}\overrightarrow{H_\perp'}} + \int_\Omega \frac{1}{\epsilon}\nabla H_z \cdot \overline{\nabla H_z'}, \\ c(\beta;\mathbf{H},\mathbf{H}') &:= i\beta \int_\Omega \frac{1}{\epsilon}\left(\nabla H_z \cdot \overline{\overrightarrow{H_\perp'}} - \overrightarrow{H_\perp}\cdot\overline{\nabla H_z'}\right) \\ &\quad + \beta^2 \int_\Omega \frac{1}{\epsilon}\overrightarrow{H_\perp}\cdot\overline{\overrightarrow{H_\perp'}}. \end{aligned}$$

Note that for $\beta \neq 0$, the functional framework and the compact part $c(\beta;\cdot,\cdot)$ depend on $\beta$, then we cannot decouple $\overrightarrow{H_\perp}$ from $H_z$ and rewrite (4) into two problems. Due to the sign-changing permittivity, the form $a$ is not coercive. Besides, coercivity can be recovered under some conditions on the ratio $\epsilon_\mathrm{d}/\epsilon_\mathrm{m}$ and the geometry of $\Sigma$, via the $\mathtt{T}$-coercivity approach. Particularly it tells us that (4) (with some data at the right-hand side) is of Fredholm type if and only if there exists an isomorphism $\mathtt{T}$ of $\mathbf{V}_T(\beta;\mu;\Omega)$ such that $a(\cdot,\mathtt{T}\cdot)$ is coercive, and $c(\beta;\cdot,\cdot)$ is compact. As mentioned above, this theory has been developed for instance in [1, 2] providing *ad hoc* operators $\mathtt{T}$ (explicit for scalar problems, abstract for Maxwell's). In that case, it has been proved that the form $a_\perp$ is in fact coercive for any value of $\epsilon < 0$, while we recover coercivity for the form $a_z$ under some conditions.

For the 2.5D case, we extend the results coupling those from scalar and Maxwell's problems. As we cannot decouple the components of $\mathbf{H}$, the construction of the operator $\mathtt{T}$ now involves an operator $\mathtt{T}_z$ from the scalar problem, an operator $\mathtt{T}_\perp$ from Maxwell 2D problems, and we add to $\mathtt{T}_\perp$ a potential solution of an elliptic problem whose right-hand side depends on $\mathtt{T}_z$ and $\beta$.

With the Riesz representation we introduce the operator $A(\beta) \in \mathcal{L}(\mathbf{V}_T(\beta;\mu;\Omega))$ associated to the form $a(\cdot,\cdot) + c(\beta;\cdot,\cdot)$. Once we have proved that the forward problem is of Fredholm type, we can prove that $A(\beta)$ is self-adjoint and has compact resolvent, so that its spectrum is composed of a sequence of positive and negative eigenvalues.

## 3 Approximation of the guided modes

Finally, following [5], and in the spirit of [4] we can provide error estimates for the approximation of the eigenvalues using edge elements. To do so, we first have to state compactness of the discrete operators and norm convergence towards $A(\beta)$: this requires to transpose $\mathtt{T}$-coercivity to the discrete problem and some conditions on the mesh [3].

**Acknowledgments**

We thank the ANR project METAMATH and the DGA for financial support.

**References**

[1] A.-S. Bonnet-Ben Dhia, L. Chesnel and P. Ciarlet Jr., T -coercivity for scalar interface problems between dielectrics and metamaterials, *Math. Mod. Num. Anal.* **46**(2012), pp. 1363-1387.

[2] A.-S. Bonnet-Ben Dhia, L. Chesnel and P. Ciarlet Jr., Two-dimensional Maxwell's equations with sign-changing coefficients, *Appl. Num. Math.* **79** (2014), pp. 29-41.

[3] L. Chesnel, P. Ciarlet Jr., T-coercivity and continuous Galerkin methods: application to transmission problems with sign changing coefficients, *Numer. Math.* **124** (2013), pp. 1-29.

[4] P. Joly, C. Poirier, J.-E. Roberts, P. Trouve, A new non-conforming finite element method for computation of electromagnetic guides waves, *SIAM J. Numer. Appl.* **33** (1996), pp. 1494-1525.

[5] J.E. Osborn, Spectral approximation for compact operators, *Math. Comp.* **29** (1975), pp. 712–725.